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THE PRESSURE BUILD-UP CURVE FOR A FRACTAL CRACKED POROUS MEDIUM. LINEAR THEORY†

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The long-term asymptotic behaviour of the pressure build-up curve is found using the fractal model of a cracked medium (CPM) [1]. All processes are assumed to be isothermal.

The motion of a fluid in a CPM is usually described by a model [2–4] in which the cracks and the matrix are represented by continua penetrating one another, each with its own individual porosity and permeability, the saturating fluid being exchanged between the cracks and the matrix. It is known, however, that in real cracked rocks the cracks form a system which cannot be regarded as continuous at distances comparable with the size of porous blocks.

The theory of fractals, i.e. sets with fractional spatial dimensions [5, 6], has been used recently to describe objects with complex irregular geometry. It has been observed that the cracks formed when the system is fractured are well described by fractal geometry [7–9]. A model of CPM was proposed in [1] in which the cracks form a fractal of Hausdorff–Birkhoff dimension d embedded in a porous continuum of spatial dimension ($D \geqslant d$, $D=2$ or $D=3$; $D=2$ corresponds to the plane problem, while $D=3$ to the spatial problem; when $d=D$ the equations of the model [1] reduce to the equations of the continuum model [2–4]).

We will adopt the usual meaning of a "fractal" used in the literature [5, 6]: fractional dimension d, local self-similarity on the average, and power asymptotic behaviour of all fractal characteristics averaged over a sphere of sufficiently large radius.

Let $\rho = \rho_1(t, r)$ be the density of the fluid inside the cracks, let $\rho_2 = \rho_2(t, r)$ be the density of the fluid in the matrix, and let t be the time and r the radial coordinate. The following conservation of mass integral equations are satisfied for the cracks and blocks, respectively

$$\frac{d}{dt}\int_{r_1\leqslant r\leqslant r_2} m_1\rho_1 d\mu_H^d = \int_{r=r_1} j_n d\mu_S^d - \int_{r=r_2} j_n d\mu_S^d + \int_{r_1\leqslant r\leqslant r_2} q d\mu_H^d \tag{1}$$

$$\frac{d}{dt}\int_{r_1\leqslant r\leqslant r_2} m_2\rho_2 d\mu_H^D = -\int_{r_1\leqslant r\leqslant r_2} q d\mu_H^d \tag{2}$$

Here r_1 and r_2 are arbitrary positive quantities, m_1 is a geometric factor characterizing the degree of opening of the cracks, m_2 characterizes the porosity of the blocks, j_n is the radial component of the mass flow from the blocks into cracks, $d\mu_H^d$ is the Hausdorff measure of a set of spatial dimension d [10], and $d\mu_S^d$ is a measure on the intersection of the fractal and a sphere of radius r defined by $d\mu_H^d = dr d\mu_S^d$. The relations

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$$\int_{r=r_1} d\mu_S^d = r_1^{d-1} \int_{r=1} d\mu_S^d = r_1^{d-1}\alpha_d, \quad \alpha_d = 2\pi^{d/2}/\Gamma(d/2) \tag{3}$$

holds, where α_d is the surface area of the $(d-1)$-dimensional unit sphere.

One can usually neglect the mass of the fluid inside the cracks. Then Eq. (1) takes the form

$$0 = \int_{r=r_0} j_n d\mu_S^d - \int_{r=r_1} j_n d\mu_S^d + \int_{r_0 \le r_1 \le r_2} q d\mu_H^d \tag{4}$$

It is natural to choose the expressions for j_n and q in the form [1]

$$j_n = -\frac{k}{\mu}\rho_1 \frac{\partial p_1}{\partial r}, \quad q = \frac{\beta}{\mu}(\rho_2 p_2 - \rho_1 p_1) r^{-\varepsilon} \tag{5}$$

Here $p_1 = p(\rho_1)$ and $p_2 = p(\rho_2)$ are the pressures inside the cracks and blocks, respectively, $\mu = \mu(\rho)$ is the shear viscosity, $\varepsilon = (D-d)$ is the space deficiency, and k and β are positive constants.

We shall assume that the rock framework is non-deformable. Then the equations

$$0 = r^{-(d-1)} \frac{\partial}{\partial r}\left(\mu^{-1} r^{d-1} \rho_1 \frac{\partial p_1}{\partial r}\right) + \frac{\beta}{k\mu}(\rho_2 p_2 - \rho_1 p_1) r^{-\varepsilon} \tag{6}$$

$$\frac{\partial \rho_2}{\partial t} = \frac{\alpha_d \beta}{\alpha_D m_2 \mu}(\rho_1 p_1 - \rho_2 p_2) r^{-2\varepsilon}$$

follow from (2)–(5).

It follows that the model contains phenomenological parameters of dimensions $[k] = L^{\varepsilon+2}$ and $[\beta] = L^{2\varepsilon}$.

When $d = D$ (the system of cracks turns into a continuous medium), k becomes the ordinary permeability and Eqs (6) can be reduced to the Barenblatt–Zheltov equations [2–4].

We will solve the problem in the cylindrically symmetric formulation $(D = 2)$. The first equation in (5) implies that the mass yield of a crack of radius r_0 per unit productive layer thickness amounts to

$$Q = \alpha_d r_0^{d-1} k (\mu^{-1} \rho_1 \partial p_1 / \partial r)|_{r=r_0} \tag{7}$$

We shall assume that the densities ρ_1 and ρ_2 become equal at spatial infinity, while j_n tends to zero. The problem consists of determining $p_1|_{r=0}$ for $t > 0$ when $Q = Q(t) = Q_0\theta(-t)$ from the system of equations (6) and the equation of state $p = p(\rho)$ ($\theta(t)$ is the Heaviside function and Q_0 is a constant).

Since we are concerned with a liquid, the linear approximation can be used. We set $\delta_i = (\rho_i - \rho_0)/\rho_0$, where ρ_0 is a constant having the dimensions of density, and assume that $|\delta_i| \ll 1$. Then Eqs (6) and (7) become

$$0 = \Delta_d \delta_1 + \beta k^{-1}(\delta_2 - \delta_1) r^{-\varepsilon}, \quad \partial\delta_2/\partial t = \gamma(\delta_1 - \delta_2) r^{-2\varepsilon} \tag{8}$$

$$Q = \zeta \partial\delta_2/\partial r|_{r=r_0} \tag{9}$$

$$\left(\gamma = \frac{\alpha_d \beta \rho_0 p_\rho(\rho_0)}{2\pi m_2 \mu(\rho_0)}, \quad \zeta = \frac{\alpha_d r_0^{d-1} k \rho_0 p_\rho(\rho_0)}{\mu(\rho_0)}, \quad \Delta_d^{(r)} = r^{-(d-1)} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r}\right)$$

When $t < 0$ problem (8), (9) has a steady-state solution

$$\delta_1 = \delta_2 = \delta_0 = \frac{Q_0 r_0^{d-1} r^{2-d}}{(2-d)\zeta} + C_1$$

Since one must determine $(\delta_1 - \delta_0)$ for $t > 0$ to find the pressure build-up curve, by the linearity of the problem it suffices to solve the system with the following initial and boundary conditions

$$\delta_1|_{t=0} = \delta_2|_{t=0} = 0, \quad (\delta_1 - \delta_2)|_{r=+\infty} = 0, \quad \left.\frac{\partial \delta}{\partial r}\right|_{r=r_0} = -Q_0 \zeta^{-1} \theta(t), \quad \left.\frac{\partial \delta_1}{\partial r}\right|_{r=+\infty} = 0 \tag{10}$$

We apply a Laplace transformation to (8) and (10) (s being the transformation parameter), substitute $r = r_0 \eta^{-1/(2\varepsilon)}$ for the independent variable, and eliminate $\delta_2(s)$. from the equations. We therefore obtain the following equation and boundary conditions for $\delta_1(s)$

$$(\Delta_{5/2}^{(\eta)} - s\varepsilon_0(s + \gamma_0 \eta)^{-1} \eta^{-\nu_2})\delta_1 = 0; \quad d\delta_1(s)/d\eta|_{\eta=1} = \zeta_0 s^{-1}, \quad d\delta/d\eta|_{\eta=+0} = 0$$

$$\varepsilon_0 = \beta/(4\varepsilon^2 k r_0^{\varepsilon}), \quad \nu_z = \varepsilon^{-1} + 3/2, \quad \gamma_0 = \gamma r^{-2\varepsilon}, \quad \zeta_0 = Q_0 r_0/(2\varepsilon\zeta) \tag{11}$$

Since the problem consists of computing the asymptotic form of $\delta_1|_{\eta=1}$ for long positive times, it suffices to determine the asymptotic behaviour of $\delta_1(s)$ for small positive s. This problem will be solved apart from a factor of the form $(1 + O(s))$.

Note that for $0 < \eta \ll s$ the asymptotic equality $\delta_1(s) = f^0$ holds where f^0 satisfies the equation and boundary condition

$$(\Delta_{5/2}^{(\eta)} - \varepsilon_0 \eta^{-\nu_2}) f^0 = 0; \quad df^0/d\eta|_{\eta=+0} = 0 \tag{12}$$

which follow from (11).

For $s \ll \eta \ll 1$ the asymptotic equality $\delta_1(s) = f^1$ holds, where f^1 satisfies the equation and boundary condition

$$(\Delta_{5/2}^{(\eta)} - s\varepsilon_0 \gamma_0^{-1} \eta^{-(\nu_2+1)}) f^1 = 0; \quad df^1/d\eta|_{\eta=1} = \zeta_0 s^{-1} \tag{13}$$

which follow from (11).

The functions f^0 and f^1 must be connected by the compatibility conditions

$$f^0|_{\eta=\eta_*} = f^1|_{\eta=\eta_*}, \quad df^0/d\eta|_{\eta=\eta_*} = df^1/d\eta|_{\eta=\eta_*} \tag{14}$$

where $\eta_* = \lambda s$ and λ is a certain finite positive quantity.

We introduce the new auxiliary notation

$$\nu_i = \nu_2 - 2 + i, \quad \tau_i = \frac{1}{2\nu_i} \ (i = 0, 1), \quad \kappa_0 = \frac{2\varepsilon_0^{1/2}}{\nu_0}, \quad \kappa_1 = \frac{2(\varepsilon_0 s)^{1/2}}{\gamma_0^{1/2} \nu_1}$$

According to [10]

$$f^0 = C_1^0 \eta^{-1/4} K_{\tau_0}(\kappa_0 \eta^{-\nu_0/2})$$

is a solution of (12) and

$$f^1 = \eta^{-1/4}(C_1^1 K_{\tau_1}(\kappa_1 \eta^{-\nu_1/2}) + C_2^1 I_{\tau_1}(\kappa_1 \eta^{-\nu_1/2}))$$

is a solution of (13). Here $I_\alpha(z)$ are MacDonald functions [10]. The constants C_1^0, C_1^1, C_2^1 must be determined from the boundary condition in (13) and the equalities (14). From (14) we obtain a linear system, solving which we obtain $C_i^1 = C_1^0 n_i$ $(i = 1, 2)$, where

$$n_1 = \nu_1^{-1} \eta_*^{-\nu_1/2} (\kappa_1 \nu_1 I_{\tau_1}'(\kappa_1 \eta_*^{-\nu_1/2}) K_{\tau_0}(\kappa_0 \eta_*^{-\nu_0/2}) - \kappa_0 \nu_0 \eta_*^{1/2} I_{\tau_1}(\kappa_1 \eta_*^{-\nu_1/2}) K_{\tau_0}'(\kappa_0 \eta_*^{-\nu_0/2}))$$

An expression for n_2 can be obtained from the previous one by substituting I_{τ_1}, I_{τ_1}' in place of K_{τ_1}, K_{τ_1}'.

As $s \to 0$ we have the following estimate, which follows from the asymptotic properties of the MacDonald functions [10]

$$n_2 / n_1 = O(\exp(-2\kappa_1 \eta_*^{-\nu_1/2})) \tag{15}$$

Computing $df^1/d\eta|_{\eta=1}$ and using the boundary conditions in (13), we determine C_1^0. Then, expanding the MacDonald functions [10] and using (15), we find the leading asymptotic form of f^1 at $s=0$ and perform an inverse Laplace transformation. Finally, we find that the pressure build-up curve in a CPM with fractal crack geometry can be described by the asymptotic formula

$$\Delta p = \rho_0 p_\rho(\rho_0) \frac{\zeta_0 K(-\tau_1)}{\nu_1 \tau_1} \left\{ \frac{t^{\tau_1}}{\Gamma(1+\tau_1)} - K(\tau_1) + K(1-\tau_1) \frac{t^{2\tau_1 - 1}}{\Gamma(2\tau_1)} \right\}$$

$$K(x) = \nu_1^{-2x} \varepsilon_0^x \gamma_0^{-x} \Gamma(1-x) / \Gamma(1+x)$$

Unlike an ordinary porous medium with logarithmic asymptotic form [4], the power asymptotic form $\Delta p = \text{const}\, t^{\tau_1}$ is characteristic of a fractal CPM. The exponent can be expressed in terms of the dimensions of the fractal $\tau_1 = (2-d)/(4-d)$. It is seen that for $1 < d < 2$ the exponent τ_1 lies in the range (0, 1/3). By determining this exponent experimentally, one can find the fractal dimension of the crack system. Taking the limit as $d \to 2$ (which is equivalent to $\tau_1 \to 0$), one can obtain from (16) the classical logarithmic formula for the pressure build-up curve, even though the derivation of (16) given here is clearly no longer valid for $d = 2$.

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